POTENZIALI

$$\begin{cases} \nabla \times \underline{E}(\underline{r},\omega) = -j\omega\,\mu\,\underline{H}(\underline{r},\omega) \quad * \\ \nabla \times \underline{H}(\underline{r},\omega) = j\omega\,\varepsilon\,\underline{E}(\underline{r},\omega) + \underline{J}(\underline{r},\omega) \\ \nabla \cdot \varepsilon\,\underline{E}(\underline{r},\omega) = \rho(\underline{r}) \\ \nabla \cdot \mu\,\underline{H}(\underline{r},\omega) = 0 \end{cases}$$

$$\downarrow$$

$$\boxed{\underline{H} = \frac{1}{\mu}\nabla \times \underline{A}}$$

$$\underline{A}^1 = \underline{A} - \nabla\psi$$

$$\underline{H}^1 = \underline{H}$$

$$* \quad \nabla \times \left(\underline{E} + j\omega\underline{A}\right) = 0$$

$$\downarrow$$

$$\underline{E} + j\omega\underline{A} = -\nabla\phi$$

$$\boxed{\underline{E} = -\nabla\phi - j\omega\underline{A}}$$

Se $\underline{A} \rightarrow \underline{A}^1$

$$\underline{E}^1 = -\nabla\phi - j\omega\underline{A}^1 \neq \underline{E}$$

ma se

$$\phi \quad \rightarrow \quad \phi^1 = \phi + j\omega\psi$$

$$\underline{E}^1 = -\nabla\phi - \cancel{j\omega\nabla}\psi - j\underline{A} + \cancel{j\omega\nabla}\psi = \underline{E}$$

$$\begin{cases} \underline{H} = \dfrac{1}{\mu}\nabla \times \underline{A} \\ \underline{E} = -\nabla\phi - j\omega\underline{A} \end{cases} \qquad \text{invarianti se} \qquad \begin{matrix} \underline{A}^1 = \underline{A} - \nabla\psi \\ \phi^1 = \phi + j\omega\psi \end{matrix}$$

Da:

$$\nabla \times \underline{H} = j\omega\varepsilon\underline{E} + \underline{J}$$

$$\nabla \times \nabla \times \underline{A} = -j\omega\varepsilon\mu\nabla\phi + k^2\underline{A} + \mu\underline{J}$$

$$\nabla \times \nabla \times \underline{A} = \nabla\nabla\cdot\underline{A} - \nabla^2\underline{A}$$

$$\nabla\nabla\cdot\underline{A} - \nabla^2\underline{A} = -j\omega\varepsilon\mu\nabla\phi + k^2\underline{A} + \mu\underline{J}$$

$$\boxed{\nabla^2\underline{A} + k^2\underline{A} = -\mu\underline{J} + \nabla\left(\nabla\cdot\underline{A} + j\omega\varepsilon\mu\phi\right)}$$

Da:

$$\nabla \cdot \varepsilon \underline{E} = \rho \qquad \rightarrow \qquad \nabla \cdot \left(-\nabla\phi - j\omega \underline{A}\right) = \frac{\rho}{\varepsilon}$$

$$-\nabla^2\phi - k^2\phi = \frac{\rho}{\varepsilon} - k^2\phi + j\omega \nabla \cdot \underline{A}$$

$$\boxed{\nabla^2\phi + k^2\phi = -\frac{\rho}{\varepsilon} - j\,\omega\left(\nabla \cdot \underline{A} + j\,\omega\,\varepsilon\,\mu\,\phi\right)}$$

Sarebbe:

$$\boxed{\begin{array}{l} \nabla^2 \underline{A} + k^2 \underline{A} = -\mu \underline{J} \\ \nabla^2\phi + k^2\phi = -\dfrac{\rho}{\varepsilon} \end{array}}$$

se

$$\boxed{\nabla \cdot \underline{A} + j\,\omega\,\varepsilon\,\mu\phi = 0}$$

<u>gauge di Lorentz</u>

se invece:

$$\nabla\cdot\underline{A}+j\omega\varepsilon\mu\phi=\chi \qquad \underline{A}^1=\underline{A}-\nabla\psi$$

$$\nabla\cdot\underline{A}^1+\nabla\cdot\nabla\psi+j\omega\varepsilon\mu\phi^1+\omega^2\varepsilon\mu\psi=\chi \qquad \phi^1=\phi+j\omega\psi$$

$$\nabla\cdot\underline{A}^1+j\omega\varepsilon\mu\phi^1=\underbrace{-\nabla^2\psi-\omega^2\varepsilon\mu\psi+\chi}_{\downarrow \atop =0} \qquad \text{se } \nabla^2\psi-\omega^2\varepsilon\mu\psi=-\chi$$

Quindi, è <u>sempre</u> possibile:

$$\boxed{\nabla\cdot\underline{A}+j\omega\varepsilon\mu\phi=0}$$

Cosa significa?

Da:

$$\nabla \cdot \underline{J} = -j\omega\rho$$

$$\frac{1}{\mu}\nabla \cdot \left(\nabla^2 \underline{A} + k^2 \underline{A}\right) = -j\omega\varepsilon\left[\nabla^2\phi + K^2\phi\right]$$

$$\underbrace{\nabla \cdot [\nabla\nabla \cdot \underline{A} - \nabla x}_{\equiv 0}\nabla x\underline{A}] + k^2\nabla \cdot \underline{A} = -j\omega\varepsilon\ \mu\,\nabla \cdot \nabla\phi - j\omega\varepsilon\ \mu\,k^2\phi$$

$$\nabla \cdot \nabla\left[\nabla \cdot \underline{A} + j\omega\varepsilon\ \mu\,\phi\right] + k^2\left[\nabla \cdot \underline{A} + j\omega\varepsilon\ \mu\,\phi\right] = 0$$

$$\downarrow$$

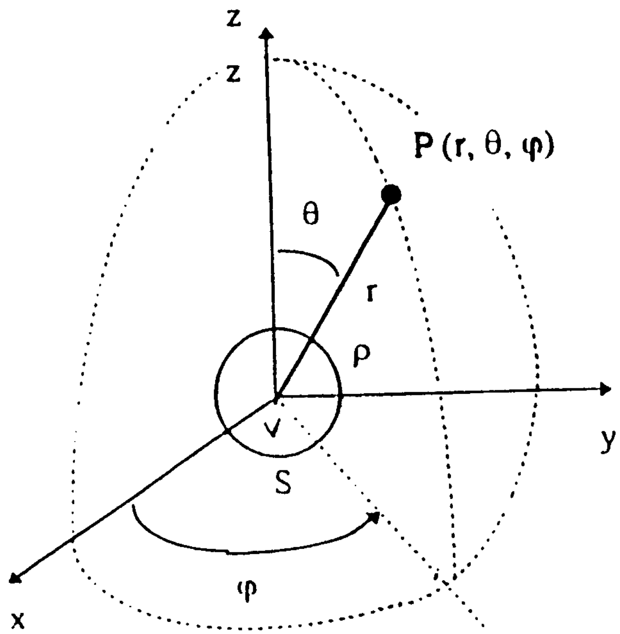
$$\boxed{\nabla \cdot \underline{A} + j\omega\varepsilon\ \mu\,\phi = 0} \qquad \phi = -\frac{\nabla \cdot \underline{A}}{j\omega\varepsilon\ \mu}$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla \phi - j\omega \mathbf{A} =$$

$$= \quad -j\omega\mathbf{A} - \frac{\nabla\nabla \cdot \mathbf{A}}{j\omega\varepsilon\ \mu} = -j\omega\left[\mathbf{A} + \frac{\nabla\nabla \cdot \mathbf{A}}{\omega^2 \varepsilon\ \mu}\right]$$

Risolviamo:

$$\nabla^2 \phi + k^2 \phi = -\frac{\rho}{\varepsilon}$$



all'esterno di V:

$$\nabla^2\phi + k^2\phi = 0$$

$$\phi = \phi(r,\theta,\varphi)$$

spazio illimitato,
mezzo omogeneo ed isotropo.

Per simmetria é :

$$\phi = \phi(r,\omega)$$

e quindi

$$\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta\phi}{\delta r}\right) + k^2\phi = 0$$

$$\phi = \frac{\phi_0(r,\omega)}{r}$$

$$\frac{1}{r^2}\frac{\delta}{\delta r}\left[r^2\frac{\delta}{\delta r}\left(\frac{\phi_0}{r}\right)\right] + k^2\frac{\phi_0}{r} = 0$$

$$\frac{1}{r^2}\frac{\delta}{\delta r}\left[r^2\left(\frac{1}{r}\frac{\delta \phi_0}{\delta r}-\frac{\phi_0}{r^2}\right)\right]+k^2\frac{\phi_0}{r}=0$$

→

$$\frac{1}{r^2}\frac{\delta}{\delta r}\left[r\ \frac{1}{\cancel{r}}\frac{\delta \phi_0}{\delta r}-\phi_0\ \right]+k^2\frac{\phi_0}{r}=0$$

$$\frac{1}{r^2}\left[\cancel{\frac{\delta \phi_0}{\delta r}}+r\frac{\delta^2 \phi_0}{\delta r^2}-\cancel{\frac{\delta \phi_0}{\delta r}}\right]+k^2\frac{\phi_0}{r}=0$$

$$\frac{1}{r}\frac{\delta^2 \phi_0}{\delta r^2}+k^2\frac{\phi_0}{r}=0$$

E’ $r\neq 0$

$$\downarrow$$

$$\frac{\delta^2 \phi_0}{\delta r^2} + k^2 \phi_0 = 0$$

$$\swarrow$$

$$\phi_0 = A\,e^{+jkr} + B\,e^{-jkr}$$

$$\downarrow$$

$$\boxed{\phi = \frac{A}{r} e^{jkr} + \frac{B}{r} e^{-jkr}}$$

<u>Condizione</u> <u>all'infinito</u>:

$$\lim_{r\to\infty} r\,\phi = c \quad \rightarrow \quad \lim_{r\to\infty}(A\,e^{+jkr} + B\,e^{-jkr}) = c$$

$$\downarrow$$

$$A = 0$$

e quindi

$$\boxed{\phi = \frac{B}{r} e^{-jkr}}$$

Condizione all'origine: la presenza di $\rho(v)$

$$\nabla^2 \phi + k^2 \phi = -\frac{\rho}{\varepsilon}$$

$$\iiint_v \nabla^2 \phi \; dV + k^2 \iiint_v \phi \, d\,V = -\iiint_v \frac{\rho}{\varepsilon} dV$$

$$\iint_s \nabla \phi \; \cdot i_n dS + k^2 \iiint_v \phi \, d\,V = -\frac{Q}{\varepsilon}$$

per $\Delta V \rightarrow 0$

$$\iint_s \nabla \phi \; \cdot i_r d\,S = -\frac{Q}{\varepsilon}$$

$$\downarrow$$

$$\nabla \phi \cdot i_r = \frac{\partial \phi}{\partial r}$$

$$\phi = \frac{B}{r} e^{-jkr}$$

$$\frac{\partial \phi}{\partial r} = -Bjk \frac{e^{-jkr}}{r} - B \frac{e^{-jkr}}{r^2}$$

e quindi:

$$\left(-Bjk \frac{e^{-jkr}}{r} - B \frac{e^{-jkr}}{r^2}\right) 4\pi r^2 = -\frac{Q}{\varepsilon}$$

$r \rightarrow 0$ $\qquad$ $e^{-jkr} \rightarrow 1$

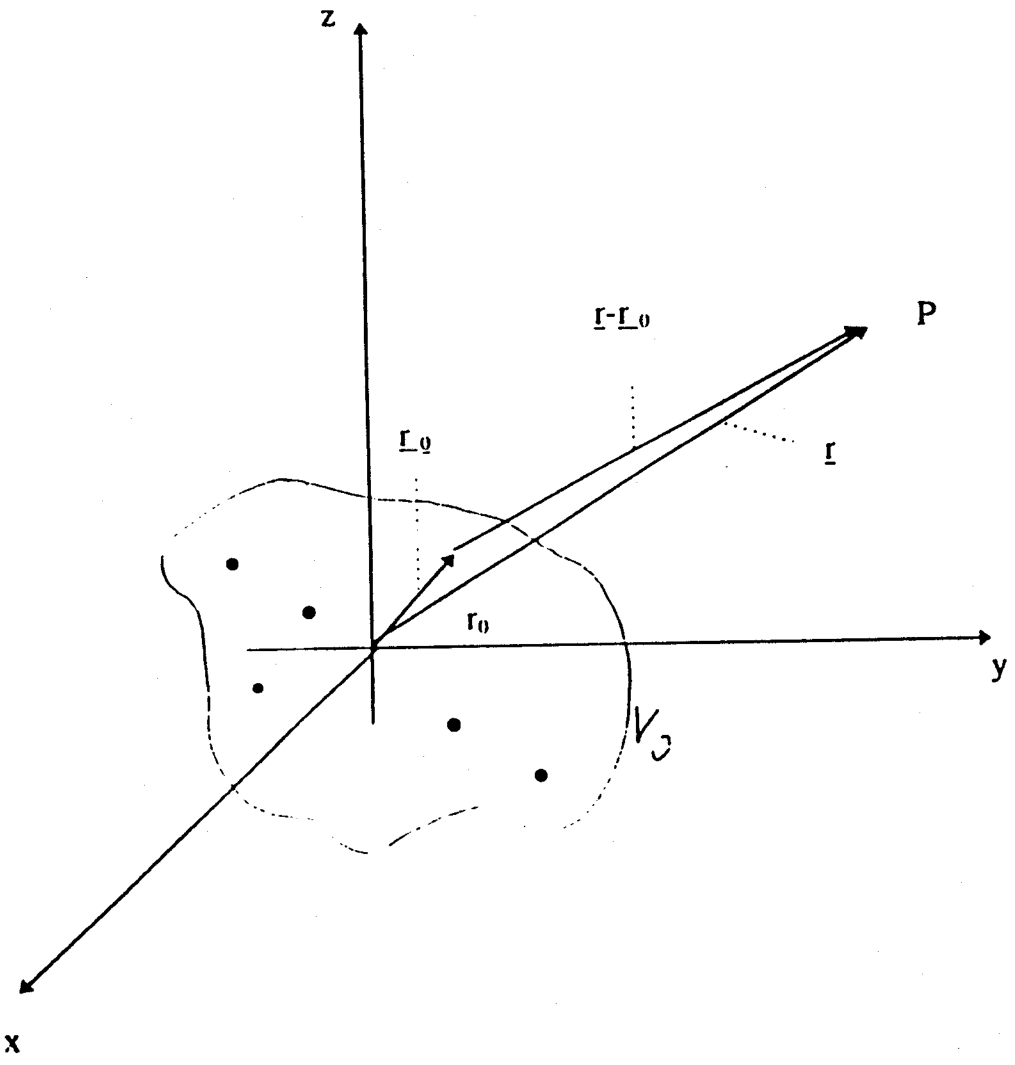
$$4\pi B = \frac{Q}{\varepsilon}$$

e quindi:

$$\boxed{\phi = \frac{Q}{4\pi\varepsilon r} e^{-jkr}}$$

Generalizzando:

$$\phi\ (\underline{r},\omega) = \frac{1}{4\,\pi\,\varepsilon}\iiint_{V_0} \rho(\underline{r}_0)\,\frac{e^{-jk|\underline{r}-\underline{r}_0|}}{|\underline{r}-\underline{r}_0|}\,d\,V_0$$



Equazione vettoriale di Helmoltz

$$\boxed{\nabla^2 \underline{A} + k^2 \underline{A} = -\mu \, \underline{J}}$$

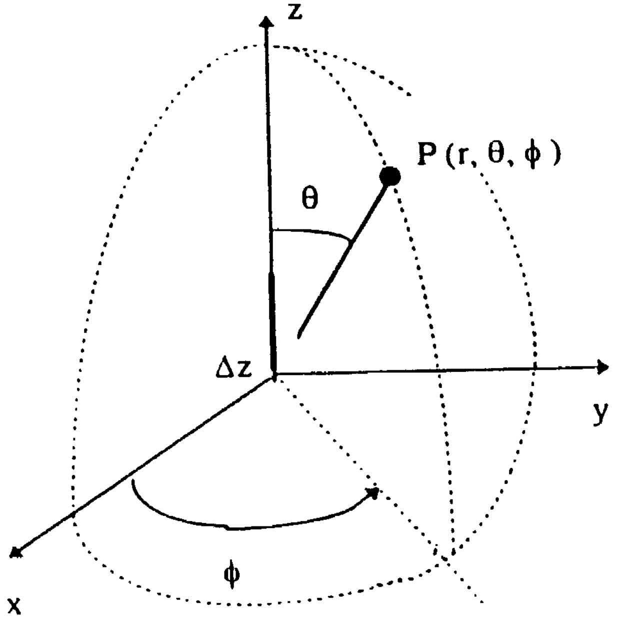
$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu \, J_x \\ \nabla^2 A_y + k^2 A_y = -\mu \, J_y \\ \nabla^2 A_z + k^2 A_z = -\mu \, J_z \end{cases}$$

$$A_i = \frac{\mu}{4\pi} \iiint_{V_0} J_i \, \frac{e^{-jk|\underline{r}-\underline{r}_0|}}{\underline{r}-\underline{r}_0} \, d\,V_0$$

e quindi

$$\underline{A} = \frac{\mu}{4\pi} \iiint_{V_0} \underline{J} \frac{e^{-jk|\underline{r}-\underline{r}_0|}}{|\underline{r}-\underline{r}_0|} d V_0$$

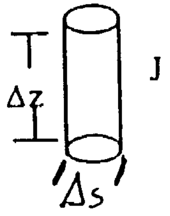
DIPOLO ELETTRICO ELEMENTARE



$I_z \qquad \Delta z << \lambda$

$$k\, z_{max} = 2\pi \frac{z_{max}}{\lambda} = 2\pi \frac{\Delta z}{2\lambda} << 2\pi$$

$I_z = J_z \Delta s = I$



$$\underline{A} = \frac{\mu}{4\pi} I \Delta z \frac{e^{-jkr}}{r} \underline{i}_z$$

in quanto

$$\nabla^2 A_x + k^2 A_x = 0$$

$$\nabla^2 A_y + k^2 A_y = 0$$

$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

ma:

$$i_z = i_r \cos\theta - i_\theta \operatorname{sen}\theta$$

$$\begin{cases} A_r = \frac{\mu}{4\pi} I\Delta z \frac{e^{-jkr}}{r} \cos\theta \\ A_\theta = -\frac{\mu}{4\pi} I\Delta z \frac{e^{-jkr}}{r} \operatorname{sen}\theta \\ A_\varphi = 0 \end{cases}$$

$$\underline{H} = \frac{1}{\mu} \nabla x \underline{A}$$

$$H_r = \frac{1}{\mu} \frac{1}{r \operatorname{sen}\theta} \left[\frac{\partial}{\partial\theta} \left(\operatorname{sen}\theta \, A_\varphi \right) - \frac{\partial A_0}{\partial\varphi} \right] = 0$$

$$H_\theta = \frac{1}{\mu}\frac{1}{r}\left[\frac{1}{\operatorname{sen}\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}\left(r A_\varphi\right)\right] = \varnothing$$

$$H_\varphi = \frac{1}{\mu}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r A_\theta\right) - \frac{\partial A_r}{\partial \theta}\right] = \frac{1}{\mu r}\left[A_\theta + r\frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right] =$$

$$= \frac{1}{\mu r}\left[A_\theta + r\frac{\mu}{4\pi} I\Delta z \operatorname{sen}\theta\left(jk\frac{e^{-jkr}}{r} + \frac{e^{-jkr}}{r^2}\right) + \frac{\mu}{4\pi} I\Delta z \frac{e^{-jkr}}{r}\operatorname{sen}\theta\right] =$$

$$= -\frac{1}{4\pi r}\cancel{I\Delta z\frac{e^{-jkr}}{r}}\operatorname{sen}\theta + \frac{1}{4\pi} I\Delta z \operatorname{sen}\theta\left(\frac{jk\,e^{-jkr}}{r} + \frac{e^{-jkr}}{r^2}\right) + \frac{1}{4\pi}\cancel{I\Delta z\frac{e^{-jkr}}{r}}\operatorname{sen}\theta$$

$$\boxed{H_\varphi = \frac{I\Delta z}{4\pi}\left(j\frac{2\pi}{\lambda r} + \frac{1}{r^2}\right)\operatorname{sen}\theta\, e^{-jkr}}$$

$$\underline{E} = -j\omega A + \frac{\nabla\nabla\cdot\underline{A}}{j\omega\varepsilon\,\mu} \qquad \zeta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\boxed{\begin{aligned} E_r &= \zeta \frac{I \Delta z}{2\pi}\left[\frac{1}{r^2} - j\frac{\lambda}{2\pi\, r^3}\right]\cos\theta\, e^{-jkr} \\ E_\theta &= \zeta \frac{I \Delta z}{4\pi}\left[j\frac{2\pi}{\lambda r} + \frac{1}{r^2} - j\frac{\lambda}{2\pi\, r^3}\right]\operatorname{sen}\theta\ e^{-jkr} \end{aligned}}$$

$$E_\varphi = 0$$

$$\frac{2\pi}{\lambda r_0} = \frac{1}{r_0^2} \qquad\qquad r_0 = \frac{\lambda}{2\pi}$$

$$r >> r_0 \qquad \begin{cases} E_\theta = j\zeta \dfrac{I \Delta z}{2\lambda r}\operatorname{sen}\theta\ e^{-jkr} \\ H_\varphi = j\dfrac{I \Delta z}{2\lambda r}\operatorname{sen}\theta\ e^{-jkr} \end{cases}$$

$$\underline{S}^e = \frac{1}{2}\underline{E} \, \mathrm{x} \, H^* = \frac{1}{2}\begin{vmatrix} i_r & i_\theta & i_\varphi \\ E_r & E_\theta & 0 \\ 0 & 0 & H_\varphi{}^* \end{vmatrix} =$$

$$= \frac{1}{2}\left(E_\theta H_\varphi{}^* i_r - E_r H_\varphi{}^* i_\theta\right)$$

$$S_r^e = \frac{1}{2}\zeta\frac{I^2\Delta\, z^2}{(4\pi)^2}\,\mathrm{sen}^2\,\theta\left[\left(\frac{2\pi}{\lambda r}\right)^2 \cancel{+\,j\frac{2\pi}{\lambda r^3}} \cancel{-\,j\frac{2\pi}{\lambda r^3}} \cancel{+\,\frac{1}{r^4}} \cancel{-\,\frac{1}{r^4}} - j\frac{\lambda}{2\pi\, r^5}\right] =$$

$$= \frac{1}{2}\zeta\frac{I^2\Delta\, z^2}{(4\pi)^2}\;\mathrm{sen}^2\,\theta\left[\left(\frac{2\pi}{\lambda r}\right)^2 - j\frac{\lambda}{2\pi\, r^5}\right]$$

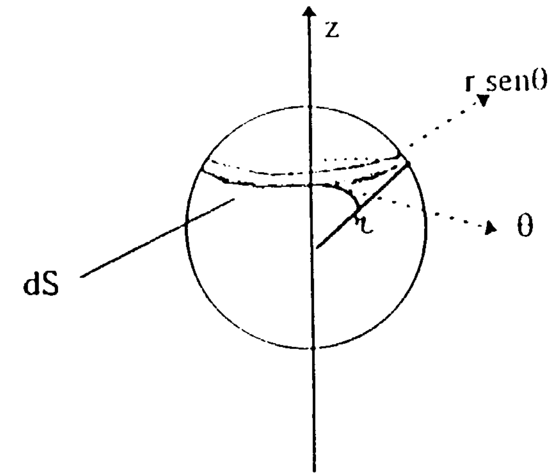
$$S_\theta^e = -\frac{1}{2}\zeta\frac{I^2\Delta\, z^2}{(4\pi)^2}\,2\,\mathrm{sen}\,\theta\,\cos\theta\left[-j\frac{2\pi}{\lambda r^3} \cancel{+\,\frac{1}{r^4}} \cancel{-\,\frac{1}{r^4}} - j\frac{\lambda}{2\pi\, r^5}\right] =$$

$$= j\frac{1}{2}\zeta\frac{I^2\Delta\, z^2}{(4\pi)^2}\,\mathrm{sen}\,2\theta\left[\frac{2\pi}{\lambda r^3} + \frac{\lambda}{2\pi\, r^5}\right]$$

$$P_r = \oiint_s S^c \cdot i_r \, dS = \oiint_s S_r^c \, dS =$$

$$= \frac{1}{2} \zeta \left(\frac{I \Delta z}{4\pi} \right)^2 \oiint_s \left[\left(\frac{2\pi}{\lambda r} \right)^2 - j \frac{\lambda}{2\pi r^5} \right] \operatorname{sen}^2 \theta \, dS =$$

$$= \frac{1}{2} \zeta \left(\frac{I \Delta z}{4\pi} \right)^2 \int_0^\pi \left[\left(\frac{2\pi}{\lambda r} \right)^2 - j \frac{\lambda}{2\pi r^5} \right] 2\pi r^2 \operatorname{sen}^3 \theta \, d\theta$$



$dS = 2\pi \, r \operatorname{sen}\theta \, r \, d\theta =$
$= 2\pi \, r^2 \operatorname{sen}\theta \, d\theta$

$$P_r = \frac{1}{2}\zeta\left(\frac{I\Delta z}{4\pi}\right)^2\left[\left(\frac{2\pi}{\lambda}\right)^2 - j\frac{\lambda}{2\pi r^3}\right]2\pi\underbrace{\int_0^\pi \mathrm{sen}^3\,\theta\ d\theta}_{4/3} =$$

$$= \frac{\pi}{3}\zeta I^2\left(\frac{\Delta z}{\lambda}\right)^2 - j\frac{1}{24\pi^2}\zeta I^2\left(\frac{\Delta z}{\lambda}\right)^2\left(\frac{\lambda}{r}\right)^3$$

DIPOLO ELETTRICO ELEMENTARE

↓

$\Delta z << \lambda$

CAMPO LONTANO

↓

$r >> r_0$

$$r_0 = \frac{\lambda}{2\pi}$$

$$E_\theta = j\zeta\frac{I\Delta z}{2\lambda r}\,\mathrm{sen}\,\theta\,e^{-jkr}$$

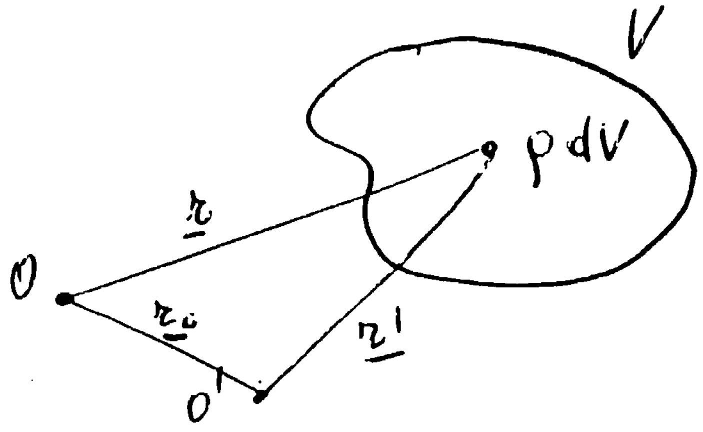
$$H_{\varphi} = j\frac{I\Delta z}{2\lambda\, r}\,\text{sen}\,\theta\; e^{-jkr}$$

$$E_{\theta} = \zeta\, H_{\varphi}$$

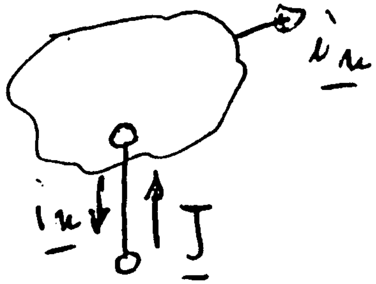
$$P_r = \oiint S_r^c\, dS = \frac{\pi}{3}\,\zeta\, I^2\left(\frac{\Delta z}{\lambda}\right)^2 - j\frac{1}{\underline{24\pi^2}}\zeta\, I^2\left(\frac{\Delta z}{\lambda}\right)^2\left(\frac{\lambda}{r}\right)^3$$

MOMENTO DI DIPOLO ELETTRICO

$$\underline{U}_e = \iiint_V \rho\, \underline{r}\, dV \qquad [\text{ Coulomb - m }]$$



$$\underline{U}_e^1 = \iiint_V \rho\, r^1\, dV = \iiint_V \rho\, \underline{r} dV + r_0 \iiint_V \rho\, dV$$



$$\nabla \cdot \underline{J} = -j\omega\rho$$

$$\iint \underline{J} \cdot i_n \, dS = -j\omega \iiint_v \rho \, dV$$

$$-I + j\omega q = 0$$

$$-I\Delta z + j\omega q \Delta z = 0 \qquad U_e = q\Delta z$$

$$\downarrow$$

$$U_e = \frac{I\Delta z}{j\omega}$$

$$\begin{cases} E_r = \dfrac{U}{2\pi\varepsilon}\left[\quad + \dfrac{j\omega}{c r^2} + \dfrac{1}{r^3}\right]\cos\theta \, e^{-jkr} \\ E_\theta = \dfrac{U}{4\pi\varepsilon}\left[-\dfrac{\omega^2}{c^2 r} + j\dfrac{\omega}{c r^2} + \dfrac{1}{r^3}\right]\text{sen}\,\theta \, e^{-jkr} \\ H_\varphi = \dfrac{U}{4\pi\varepsilon\zeta}\left[-\dfrac{\omega^2}{c^2 r} + j\dfrac{\omega}{c r^2} \quad \right]\text{sen}\,\theta \, e^{-jkr} \end{cases}$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

per $\omega \to 0$ $\quad I \to 0$ ma $U \neq 0$

quindi: <u>espressioni valide anche nel caso statico</u>

TEOREMA DI BABINET

$\underline{E}_1 \to H_2 \qquad \underline{H}_1 \to -\underline{E}_2 \qquad \varepsilon \underset{\leftarrow}{\to} \mu$

$\underline{J}_1 \to \underline{J}m_2 \qquad \rho_1 \to \rho m_2$

Se $\underline{E}_1$, $\underline{H}_1$ sostenuto da $\underline{J}_1$, ρ_1 è soluzione di Maxwell, lo è anche $\underline{E}_2$, $\underline{H}_2$ sostenuto da Jm_2, ρm_2

<u>Condizioni al contorno:</u>

se S è un c.e.p. $(E_{tg} = 0)$, dopo, S è un c.m.p. $(H_{tg} = 0)$

$$\nabla \times \underline{E}_1 = -j\omega\,\mu\,\underline{H}_1 \quad \rightarrow \quad \nabla \times \underline{H}_2 = j\omega\varepsilon\,\underline{E}_2$$

$$\nabla \times \underline{H}_1 = j\omega\varepsilon\,\underline{E}_1 + \underline{J}_1 \qquad \nabla \times \underline{E}_2 = -j\omega\,\mu\,\underline{H}_2 - J_{m2}$$

$$\nabla \cdot \mu\,\underline{H}_1 = 0 \qquad \nabla \cdot \mu\,\underline{H}_2 = \rho_{m2}$$

$$\nabla \cdot \varepsilon\,\underline{E}_1 = \rho_1 \qquad \nabla \cdot \varepsilon\,\underline{E}_2 = 0$$

Analogamente:

$$\nabla^2 \underline{A}_m + K^2 \underline{A}_m = -\varepsilon\,\underline{J}_m$$

$$\nabla^2 \phi_m + k^2 \phi_m = -\frac{\rho_m}{\mu}$$

$$\begin{cases} E = -\frac{1}{\varepsilon} \nabla \times A_m \\ H = -j\omega A_m - \nabla \phi_m \end{cases}$$

$$I_m \Delta z = j\omega U_m$$

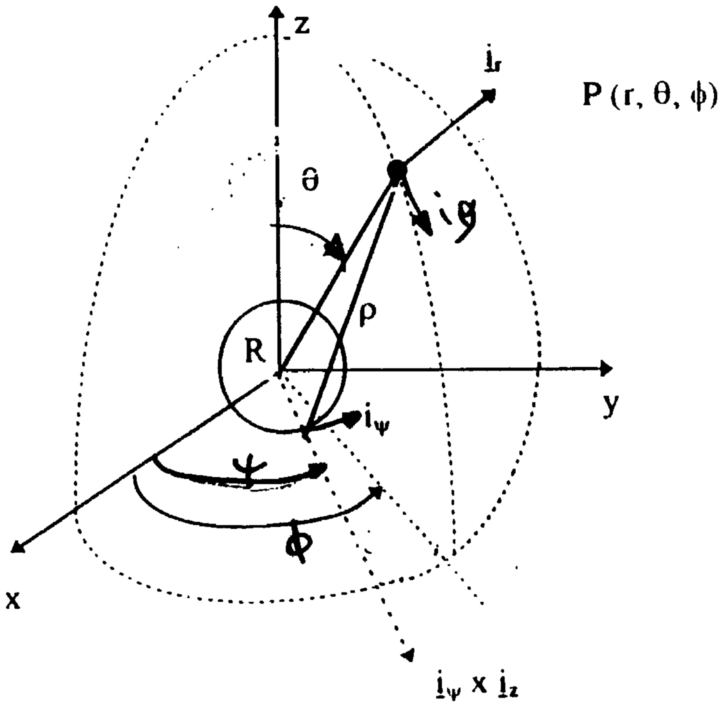
$$\underline{U'} \rightarrow \underline{U}_m$$

$$H_r = \frac{U_m}{2\pi \mu} \left[\qquad \frac{j\omega}{c r^2} + \frac{1}{r^3} \right] \cos\theta \, e^{-jkr}$$

$$H_\theta = \frac{U_m}{4\pi \mu} \left[\qquad \qquad \right] \operatorname{sen}\theta \, e^{-jkr}$$

$$E_\varphi = -\frac{\zeta U_m}{4\pi \mu} \left[\qquad \qquad \right] \operatorname{sen}\theta \, e^{-jkr}$$

Spira



Si è visto che il **potenziale** vettore di un elemento di corrente è:

$$\underline{A} = \frac{\mu}{4\pi} I \Delta z \frac{e^{-jkr}}{r} i_z$$

l'elemento della spira vale: $\boxed{Rd\psi}$

$$\underline{i}_{\psi} = -\underline{i}_{x}\,\mathrm{sen}\psi + \underline{i}_{y}\cos\psi$$

E’ inoltre:

$$2\pi R \ll \lambda, \quad I = \mathrm{cost.}$$

e quindi:

$$d\underline{A} = \frac{\mu}{4\pi} I R\, d\psi \frac{e^{-j\beta\rho}}{\rho} \underline{i}_{\psi} \qquad \boxed{k = \beta}$$

$$\boxed{\underline{A} = \frac{\mu}{4\pi} I R \int_{0}^{2\pi} \frac{e^{-j\beta\rho}}{\rho} \underline{i}_{\psi} d\psi}$$

$$\frac{e^{-j\beta\rho}}{\rho} = -j\beta\sum_{0}^{\infty}{}_{n}(2n+1)P_{n}(\cos\alpha)j_{n}(\beta R)h_{n}^{(2)}(\beta r)$$

$P_n(\cos\alpha)$: polinomi di Legendre di arg. $\underline{\cos\alpha}$
$j_n(\beta R)$: funzioni sferiche di Bessel di arg. $\underline{\beta R}$
$h_n^{(2)}(\beta r)$: funzioni sferiche di Hankel di arg. $\underline{\beta r}$

Solo $\cos\alpha$ è funzione di ψ, in quanto: $\cos\alpha = \operatorname{sen}\theta\cos(\phi - \psi)$

$$\underline{A} = -\frac{\mu}{4\pi} IR\, j\beta\sum_{0}^{\infty}{}_{n}(2n+1)j_{n}(\beta R)h_{n}^{(2)}(\beta r)\cdot\int_{0}^{2\pi}P_{n}(\cos\alpha)i_{\psi}d\psi$$

$P_0(\cos\alpha) = 1$ $\qquad$ $P_1(\cos\alpha) = \cos\alpha$

$$\rightarrow \int_{0}^{2\pi}P_{0}(\cos\alpha)i_{\psi}d\psi = \int_{0}^{2\pi}i_{\psi}d\psi = -i_{x}\int_{0}^{2\pi}\operatorname{sen}\psi\, d\psi + i_{\psi}\int_{0}^{2\pi}\cos\psi\, d\psi = 0$$

$$\rightarrow \int_{0}^{2\pi}P_{1}(\cos\alpha)i_{\psi}d\psi = \int_{0}^{2\pi}\operatorname{sen}\theta\cos(\phi-\psi)i_{\psi}d\psi =$$

$$= \operatorname{sen}\theta\int_{0}^{2\pi}\cos(\phi-\psi)(-i_{x}\operatorname{sen}\psi + i_{y}\cos\psi)d\psi = \pi\operatorname{sen}\theta\, i_{\phi}$$

i termini della serie: $\underline{A} = \underline{A}_0 + \underline{A}_1 + \underline{A}_2 + \ldots$

per $n = 0$ $\quad \underline{A}_0 = 0$

per $n = 1$ $\quad \int_0^{2\pi} P_1(\cos\alpha)\, i_\psi\, d\psi = \pi \operatorname{sen}\theta\, i_\phi$

$$J_n(\beta R) \approx \frac{2^n n!}{(2n+1)!}(\beta R)^n$$

per $R \to 0$ $\quad \beta R \to 0$ $\quad \boxed{J_1(\beta R) \approx \frac{1}{3}\beta R}$

$$h_1^2(\beta r) = -\frac{e^{-j\beta r}}{\beta r}\left(1 + j\frac{1}{\beta r}\right)$$

$$\underline{A}_1 = -\frac{\mu}{4\pi} I R j\beta\cancel{3}\ \frac{1}{\cancel{3}}\cancel{\beta} R(-)\frac{e^{-j\beta r}}{\cancel{\beta} r}\left(1 + j\frac{1}{\beta r}\right)\pi \operatorname{sen}\theta\, i_\phi$$

$$\underline{A}_1 = \frac{\mu}{4\pi r}\pi R^2 I j\beta\, e^{-j\beta r}\left(1 + j\frac{1}{\beta r}\right)\operatorname{sen}\theta\, i_\phi$$

$\underline{A} = \underline{A}_1 + \ldots\ldots$ trascurabili se $\beta R <<$

$A_r = 0 \quad A_\theta = 0 \quad A_\phi = A_1$

$$\underline{H} = \frac{1}{\mu} \nabla x \underline{A}$$

$$H_r = \frac{1}{\mu} \frac{1}{r \operatorname{sen} \theta} \left[\frac{\theta}{\partial \theta} \left(\operatorname{sen} \theta \, A_\varphi \right) - \cancel{\frac{\partial A_\theta}{\partial \varphi}} \right] =$$

$$= \frac{1}{\mu} \frac{1}{r \operatorname{sen} \theta} \frac{\partial}{\partial \theta} \left[\underbrace{B\left(\frac{1}{r}, \frac{1}{r^2} \right)} \operatorname{sen}^2 \theta \right] = \boxed{\frac{1}{\mu r} 2B \cos\theta}$$

$$\downarrow \qquad \qquad \downarrow$$

$$B = \mu \frac{\pi R^2 I}{4\pi r} j \beta \, e^{-j\beta} \left(1 + j \frac{1}{\beta r} \right) \qquad F\left(\frac{1}{r^2}, \frac{1}{r^3} \right)$$

$$H_\theta = \frac{1}{\mu} \frac{1}{r} \left[\cancel{\frac{1}{\operatorname{sen} \theta} \frac{\partial A_r}{\partial \varphi}} - \frac{\partial}{\partial r} \left(r A_\varphi \right) \right]$$

$$H_\phi = \frac{1}{\mu} \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r A_\theta \right) - \frac{\partial Ar}{\partial \theta} \right] = 0$$

$$H_\theta = -\frac{1}{\mu\, r}\frac{\partial}{\partial r}\left(rA_\phi\right) = -\frac{1}{\mu\, r}\frac{\mu}{4\pi}\pi\, R^2 Ij\beta\ \text{sen}\,\theta \cdot$$

$$\cdot\left\{e^{-j\beta r}\left[-\frac{j}{\beta\, r^2}\right]+\left[1+j\frac{1}{\beta\, r}\right]\left(-j\beta\right)e^{-j\beta r}\right\} =$$

$$\left(\text{trascurando}\quad \frac{1}{r^2}\quad ,\quad \frac{1}{r^3}\right)$$

$$= \frac{-\pi\, R^2 I\,\beta}{4\pi\, r}\frac{2\pi}{\lambda}\text{sen}\,\theta\, e^{-j\beta r} =$$

$$= -\frac{\pi\, R^2\,\beta\, I}{2\lambda\, r}\text{sen}\,\theta\, e^{-j\beta r}$$

$$E_\phi = -\zeta H_\theta \qquad S^{sp} = \begin{vmatrix} i_r & i_\theta & i_\phi \\ 0 & 0 & E_\phi \\ H_r^* & H_\theta^* & 0 \end{vmatrix}$$

$$\underset{\sim}{S}^{sp} = \frac{1}{2}\underset{\sim}{E} \times H^* = \frac{1}{2}\left(\underbrace{E_\phi H_r^* i_\theta}_{S_\theta^{sp}} - \underbrace{E_\phi H_\theta^* i_r}_{S_r^{sp} = \zeta H_\theta H_\theta^*}\right)$$

$$P_r^{sp} = \iint S_r^{sp} \, dS = \zeta \frac{4\pi}{3} I^2 \left(\frac{\pi \Delta S}{\lambda^2}\right)^2 \qquad \Delta S = \pi R^2$$

Avevamo trovato:

$$P_r^e = \zeta \frac{4\pi}{3} I^2 \left(\frac{\Delta z}{2\lambda}\right)^2$$

$$\frac{P_r^{sp}}{P_r^e} = \left(\frac{\pi \Delta S}{\lambda^2}\right)^2 \left(\frac{2\lambda}{\Delta z}\right)^2 \qquad \text{se } \Delta z = 2\pi R$$

$$\frac{P_r^{sp}}{P_r^e} = \left(\frac{\pi R}{\lambda}\right)^2 \qquad R \lll \lambda \qquad P_r^{sp} \lll P_r^e$$

d.e.c.	d.m.c.	spira
$E_\theta = j\frac{\zeta I\Delta z}{2\lambda r} \operatorname{sen}\theta\, e^{-j\beta r}$	$H_\theta = j\frac{1}{\zeta}\frac{Im\Delta z}{2\lambda r} \operatorname{sen}\theta\, e^{-j\beta r}$	$H_\theta = \frac{-\pi R^2 \beta I}{2\lambda r} \operatorname{sen}\theta\, e^{-j\beta r}$
$H_\phi = \frac{E_\theta}{\zeta}$	$E_\varphi = -\zeta H_\theta$	$E_\varphi = \zeta H_\theta$
$P_r^e = \frac{4}{3}\pi \zeta I^2 \left(\frac{\Delta z}{2\lambda}\right)^2$	$P_r^m = \frac{4}{3}\pi \frac{1}{\zeta} I^2 m \left(\frac{\Delta z}{2\lambda}\right)^2$	$P_r^{sp} = \frac{4}{3}\pi \zeta I^2 \left(\frac{\pi \Delta S}{\lambda^2}\right)^2$

d.m.c. ≡ spira   se $\frac{1}{\zeta} Im\,\Delta z = -\pi R^2 \beta I$

↓

Momenti dipolari:

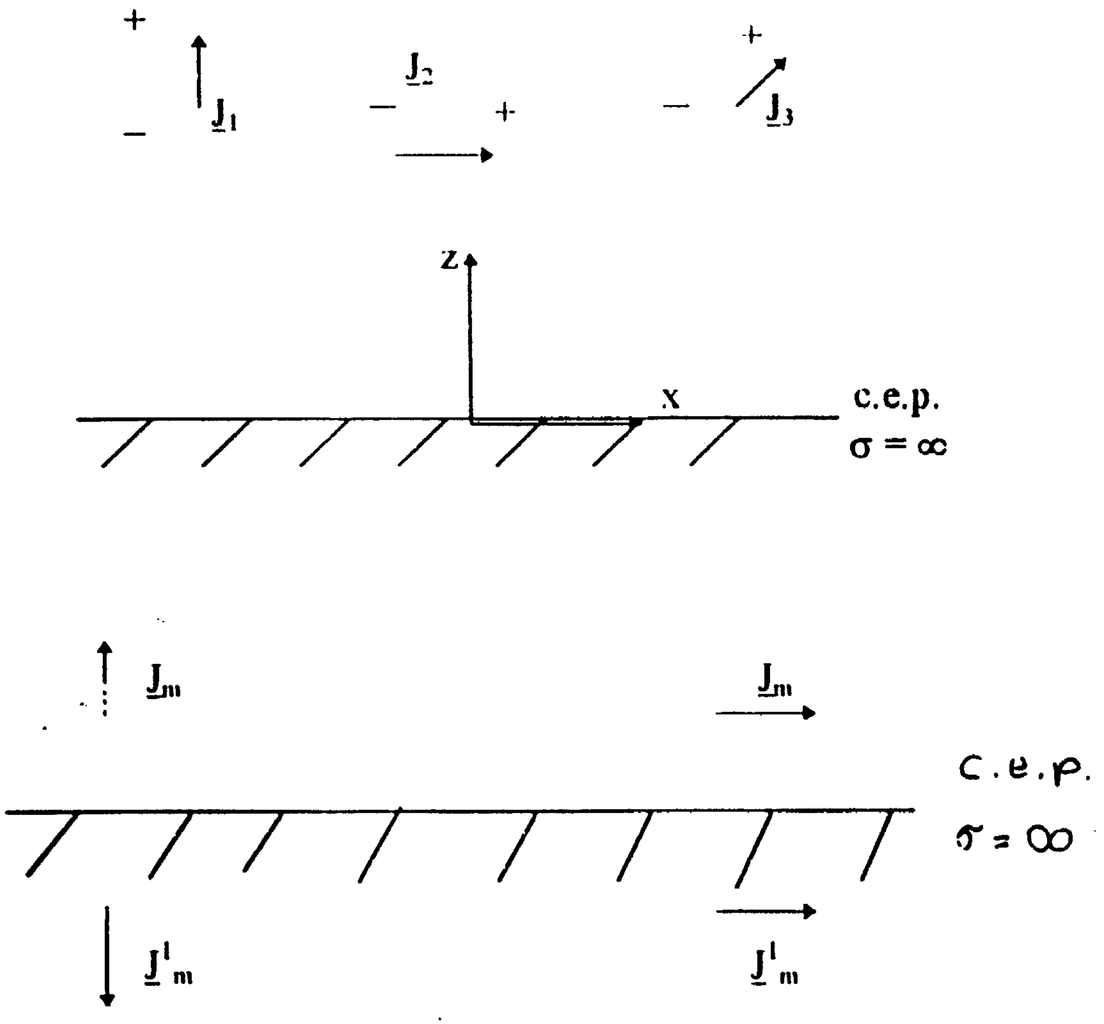
$I \Delta z = j\omega U_e$

$$\begin{tikzcd} I_m \Delta z = j\omega U_m \arrow[r] & \frac{1}{\zeta} j\omega U_m = -\pi R^2 \beta I \arrow[d] \\ & U_m = -\frac{1}{j\omega} \sqrt{\frac{\mu}{\varepsilon}} \pi R^2 \omega \sqrt{\varepsilon \mu} I = j\pi \mu R^2 I = j\mu S I \end{tikzcd}$$

<u>Principio di equivalenza di Ampère:</u>

Una spira elementare di corrente è equivalente a un dipolo magnetico, ortogonale al piano della spira; l'intensità è pari a **μ S I**, in generale N μ S I.

Teorema delle immagini



Teorema di reciprocità

$\underline{J}_1 \qquad \underline{J}_{m1} \qquad \rightarrow \qquad \underline{E}_1 \, , \, \underline{H}_1$

$\underline{J}_2 \qquad \underline{J}_{m2} \qquad \rightarrow \qquad \underline{E}_2 \, , \, \underline{H}_2$

$$+)\begin{cases} \nabla \times E_1 = -j\omega\,\mu\,H_1 - J_{m1} & \cdot \underline{H}_2 \\ \nabla \times \underline{H}_1 = j\omega\,\varepsilon\,E_1 + J_1 & \cdot \underline{E}_2 \end{cases}$$

-) $\qquad\qquad\qquad\qquad$ ε . μ

$$+)\begin{cases} \nabla \times E_2 = -j\omega\,\mu\,H_2 - J_{m2} & \cdot \underline{H}_1 \\ \nabla \times \underline{H}_2 = j\omega\,\varepsilon\,E_2 + J_2 & \cdot \underline{E}_1 \end{cases}$$

$$\underline{H}_2 \cdot \nabla \times E_1 + E_2 \cdot \nabla \times H_1 - H_1 \cdot \nabla \times E_2 - E_1 \cdot \nabla \times \underline{H}_2 =$$

$$\left(\underline{H}_2 \cdot \nabla \times \underline{E}_1 - \underline{E}_1 \cdot \nabla \times \underline{H}_2\right) + \left(\underline{E}_2 \cdot \nabla \times \underline{H}_1 - \underline{H}_1 \cdot \nabla \times \underline{E}_2\right)$$

$$\nabla \cdot \left(\underline{E}_1 \times H_2\right) + \nabla \cdot \left(\underline{H}_1 \times \underline{E}_2\right) = \ldots\ldots$$

$$\nabla \cdot \left(\underline{E}_1 \times H_2\right) - \nabla \cdot \left(\underline{E}_2 \times \underline{H}_1\right) = \ldots\ldots$$

__integrando__ = (su un volume V)

$$\iint_s \left(\underline{E}_1 x \underline{H}_2 - \underline{E}_2 x \underline{H}_1\right) \cdot i_n dS =$$

$$= \iiint_v \left(\underline{E}_2 \cdot \underline{J}_1 - \underline{H}_2 \cdot J_{m1}\right) dV - \iiint_v \left(\underline{E}_1 \cdot \underline{J}_2 - \underline{H}_1 \cdot \underline{J}_{m2}\right) dV$$

$$\uparrow \qquad \uparrow$$

integrali di reazione

$$\iint_s (\quad) = 0 \quad \left\langle \begin{array}{l} \text{a) se S è c.e.p.} \\ \text{b) se S è c.m.p.} \\ \text{c) se S è all'}\infty \end{array} \right.$$

infatti:

a)

$$\underline{E}_1 \perp S \rightarrow \underline{E}_1 x \underline{H}_2 \cdot \underline{i}_n = i_n x \underline{E}_1 \cdot \underline{H}_2 = 0$$

$$\underline{E}_2 \perp S \rightarrow \underline{E}_2 x \underline{H}_1 \cdot \underline{i}_n = i_n x \underline{E}_2 \cdot \underline{H}_1 = 0$$

b)

$$\underline{H}_1 \perp S \qquad\qquad = 0$$

$$\underline{H}_2 \perp S \qquad\qquad = 0$$

c) condizione di radiazione all'∞

$$\begin{tikzcd}
 & \boxed{\lim_{r\to\infty}\left(\underline{E}(r)-\zeta\,\underline{H}(r)\times\underline{i}_r\right)=0} \arrow[d] & \\
 & \underline{E}(r)=\zeta\,\underline{H}(r)\times\underline{i}_r \arrow[dl] \arrow[dr] & \\
\underline{E}_1(r)=\zeta\,\underline{H}_1(r)\times\underline{i}_r & ; & \underline{E}_2(r)=\zeta\,\underline{H}_2(r)\times\underline{i}_r
\end{tikzcd}$$

$$\boxed{\iint\left(\underline{E}_1\times\underline{H}_2-\underline{E}_2\times\underline{H}_1\right)\cdot\underline{i}_n\,dS=0}$$

$$\underline{i}_n\equiv\underline{i}_r$$

$$\left(\underline{E}_1 x \underline{H}_2 - \underline{E}_2 x \underline{H}_1\right) \cdot i_r = \zeta\left(\underline{H}_1 x i_r \, x \underline{H}_2 - \underline{H}_2 x i_r x \underline{H}_1\right) \cdot i_r$$

$$= \zeta\left[i_r x\left(\underline{H}_2 x \underline{H}_1\right)\right] \cdot i_r$$

$$A x\left(\underline{B} x \underline{C}\right) - C x\left(\underline{B} x \underline{A}\right) = \underline{B} x\left(\underline{A} x \underline{C}\right)$$

ma $\quad \underline{H}_2 \; x \; \underline{H}_1 \rightarrow \quad \underline{i}_r$

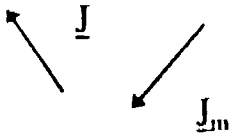
$$\zeta\,[\,\underline{i}_r \, x \, \underline{i}_r\,] \cdot \underline{i}_r = 0$$

Quindi:

$$\iiint_v\left(E_2 \cdot \underline{J}_1 - \underline{H}_2 \cdot \underline{J}_{m1}\right) dV = \iiint_v\left(\underline{E}_1 \cdot \underline{J}_2 - \underline{H}_1 \cdot \underline{J}_{m2}\right) dV$$

le reazioni sono uguali.

Sorgenti di prova (test sources)



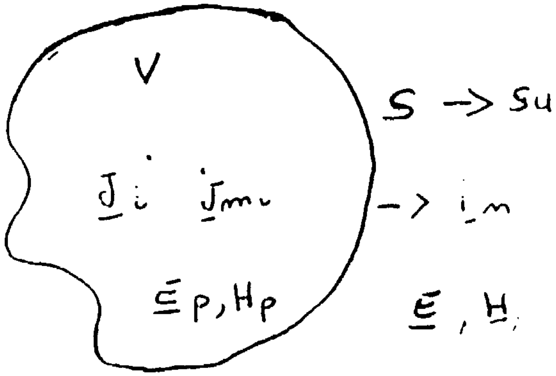
$$\iiint_V \left(\mathbf{E}_t \cdot \underline{J} - \underline{H}_t \cdot J_m\right) dV = \iiint_V \underline{E} \cdot i_q \delta(Q) dV = \quad \begin{array}{l} \underline{J}_1 = \underline{J} \\ \underline{J}_{m1} = \underline{J}_m \\ \underline{J}_2 = i_q \delta(Q) \\ \underline{J}_{m2} = 0 \\ \underline{E}_2 = \underline{E}_t \\ \underline{H}_2 = \underline{H}_t \\ \underline{E}_1 = \underline{E} \end{array}$$

$$= \underline{E}(\underline{i}_q)$$

Noti: $\underline{E}_t$, $\underline{H}_t$: prodotti da un d.e.c.,

Se si vuole $\underline{H}$, la <u>sorgente di prova</u> è $\underline{J}_{m2} = \underline{i}_q \, \delta \, (Q)$

Teorema di equivalenza



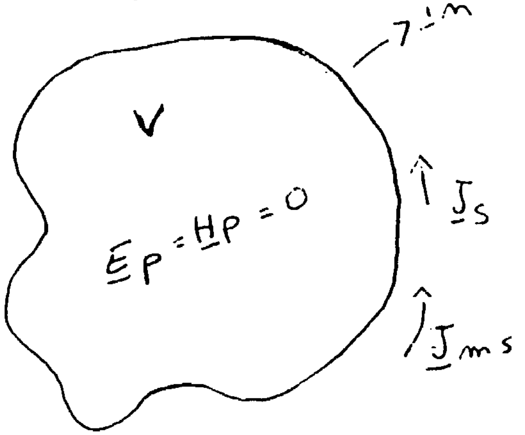
S -> superficie ideale

condizione su S

$$i_n \times \underline{H} = i_n \times \underline{H}_p$$

$$i_n \times \underline{E} = i_n \times \underline{E}_p$$

Situazione equivalente:



$\underline{E}, \underline{H}$ (come prima)

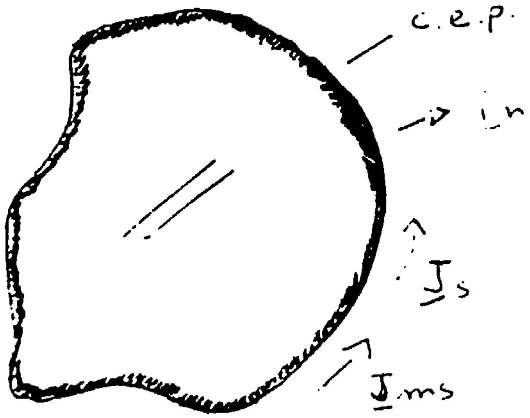
$$i_n \times \underline{H} = \underline{J}_s$$

$$-i_n \times \underline{E} = \underline{J}_{ms}$$

$$\begin{cases} \underline{J}_s = i_n \times \underline{H}_p \\ \underline{J}_{ms} = -i_n \times \underline{E}_p \end{cases}$$

Se $\underline{E}_p = \underline{H}_p = 0$

possiamo inserire in V <u>c.e.p.</u> e <u>c.m.p.</u>, senza modificare il campo (esterno) E , H.



$\underline{E}$, $\underline{H}$ sono gli stessi però

$\underline{J}_s$, $\underline{J}_{ms}$ irradiano in presenza

di un c.e.p.

Il contributo di un campo dovuto

a $\underline{J}_s$ è nullo.

Chiamamo:

$$\underline{J}_s \rightarrow \begin{matrix} \underline{E}^{I} \\ \underline{H}^{I} \end{matrix} \qquad \qquad \underline{J}_{ms} \rightarrow \begin{matrix} \underline{E}^{II} \\ \underline{H}^{II} \end{matrix}$$

Applichiamo il teorema di reciprocità:

$$\iiint_V \left(E_2 \cdot \underline{J}_1 - \underline{H}_2 \cdot \underline{J}_{m1} \right) dV = \iiint_V \left(\underline{E}_1 \cdot \underline{J}_2 - \underline{H}_1 \cdot \underline{J}_{m2} \right) dV$$

Poniamo:

$$\left.\begin{matrix} \underline{J}_1 = \underline{J}_s \\ \underline{J}_{m1} = 0 \end{matrix}\right. \rightarrow \underline{E}_1 , \underline{H}_1 \qquad \qquad \left.\begin{matrix} \underline{J}_2 = 0 \\ \underline{J}_{m2} = \underline{J}_{ms} \end{matrix}\right. \rightarrow \underline{E}_2 , \underline{H}_2$$

$$\iiint_v \underline{E}^{1} \cdot \underline{J}_s dV = -\iiint_v \underline{H}^1 \cdot \underline{J}_{ms} dV \quad \rightarrow \text{ componenti superficiali}$$

$$\begin{tikzcd} \iint_s \underline{E}^{1} \cdot \underline{J}_s dS = -\iint_s \underline{H}^1 \cdot \underline{J}_{ms} dS \arrow[d] \\ \varnothing \rightarrow \iint_S \underline{H}^1 \cdot \underline{J}_{ms} dS \rightarrow \underline{H}^1_{tg} = \end{tikzcd}$$

inoltre è $\underline{E}_{tg} = 0$ (c.e.p.)

per cui se

$$\begin{tikzcd}[column sep=small] \underline{E}^1_{tg} = 0 \arrow[dr] & \\ & \text{è} \; \boxed{E^i \,.\, H^i = 0} \quad \text{all'esterno di S} \\ \underline{H}^1_{tg} = 0 \arrow[ur] & \end{tikzcd}$$

e quindi

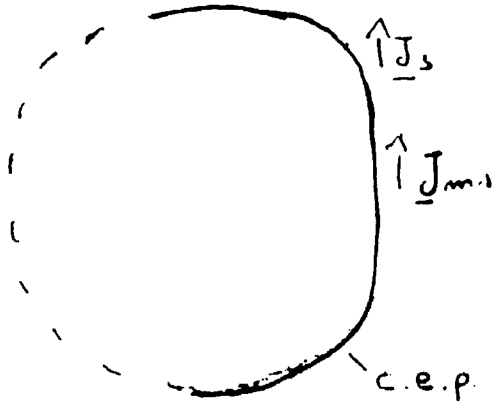
$$\left.\begin{array}{l}\underline{E} = \underline{E}'' \\ \underline{H} = \underline{H}''\end{array}\right\} \rightarrow \underline{J}_{ms} \text{ irradianti nella immediata vicinanza di c.e.p.}$$

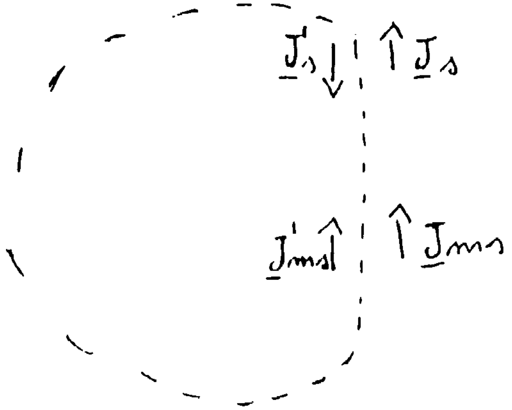
Analogamente se S è $\underline{c.m.p.}$

$$\left.\begin{array}{l}\underline{E} = E^{i} \\ \underline{H} = H^{i}\end{array}\right\} \rightarrow \underline{J}_{s} \text{ irradianti nella immediata vicinanza di c.m.p.}$$

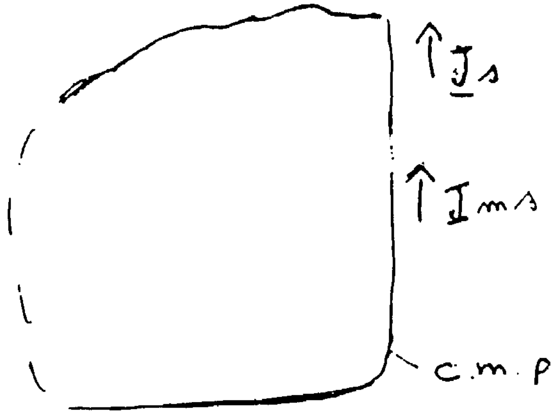
Quanto valgono $\underline{J}_s$ e $\underline{J}_{ms}$ nelle nuove condizioni?

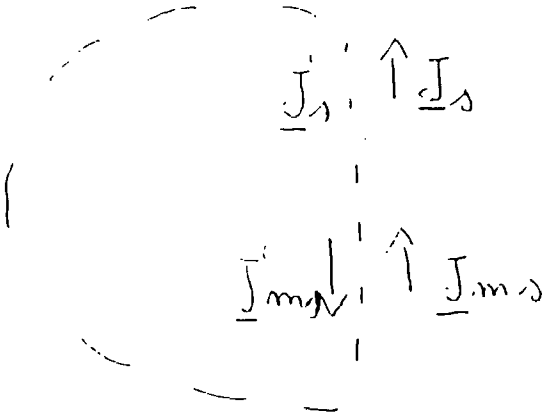
Se S fosse piana, teorema delle immagini:





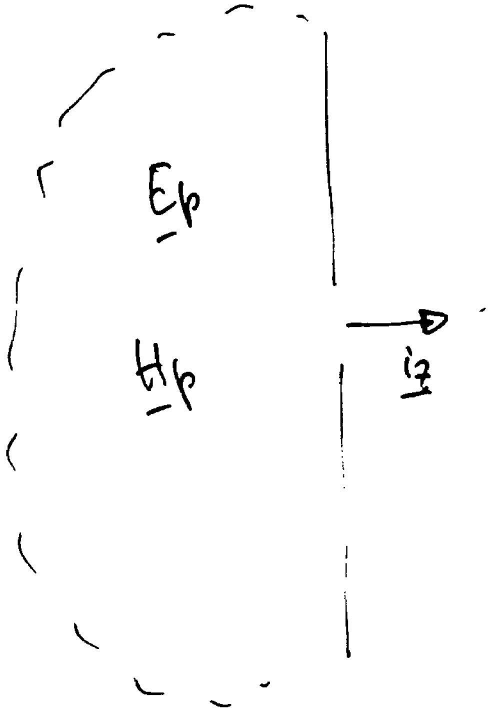
$$\underline{J}_{ms} = -2\,\underline{i}_n \times \underline{E}_p$$





$$\underline{J}_s = 2\,\underline{i}_n \times \underline{H}_p$$

Antenne ad apertura



$\underline{E}, \underline{H}$, in base alla distribuzione di correnti fittizie sull'apertura

↓

campo primario sull'apertura

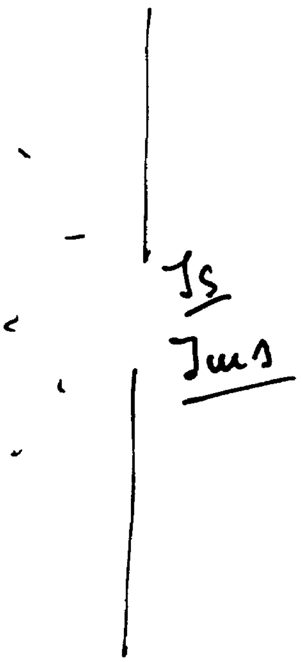
- approssimazione di Bethe

  $d \ll \lambda$

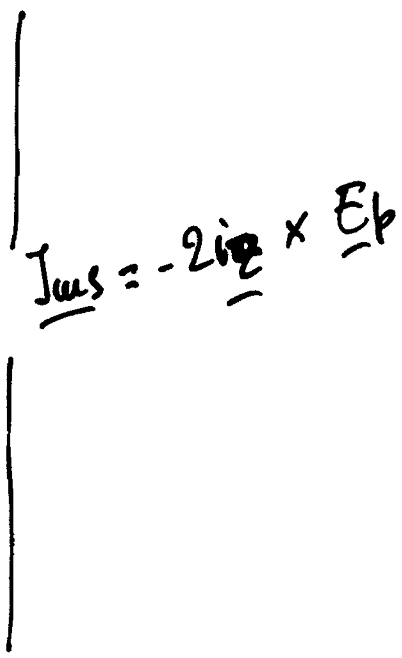
- approssimazione di Kirchoff

  $d \gg \lambda$

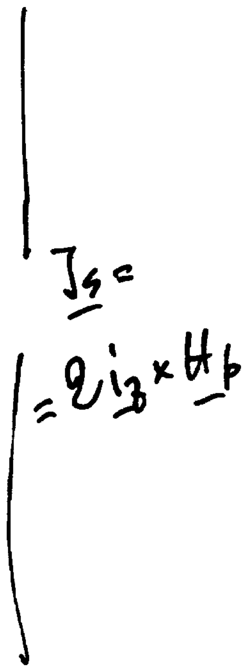
(46 ter)


$\underline{J_s}$
$\underline{J_{ms}}$

a)

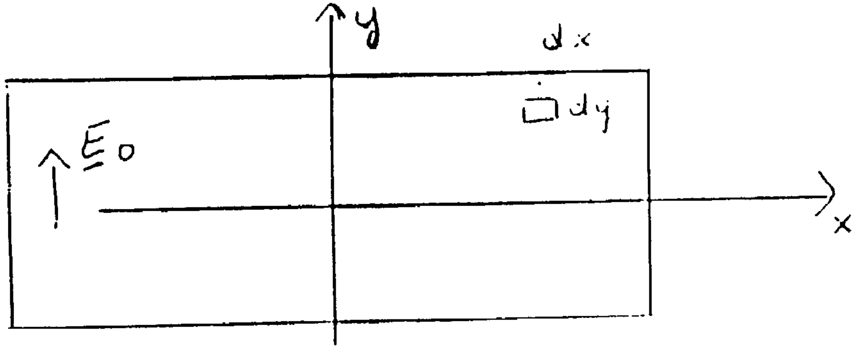

$\underline{J_{ms}} = -2\underline{i_z} \times \underline{E_p}$

b)


$\underline{J_s} = 2\underline{i_z} \times \underline{H_p}$

c)

Apertura rettangolare



$$\underline{J}_{ms} = 2\, \underline{E}_0 \times \underline{i}_z$$

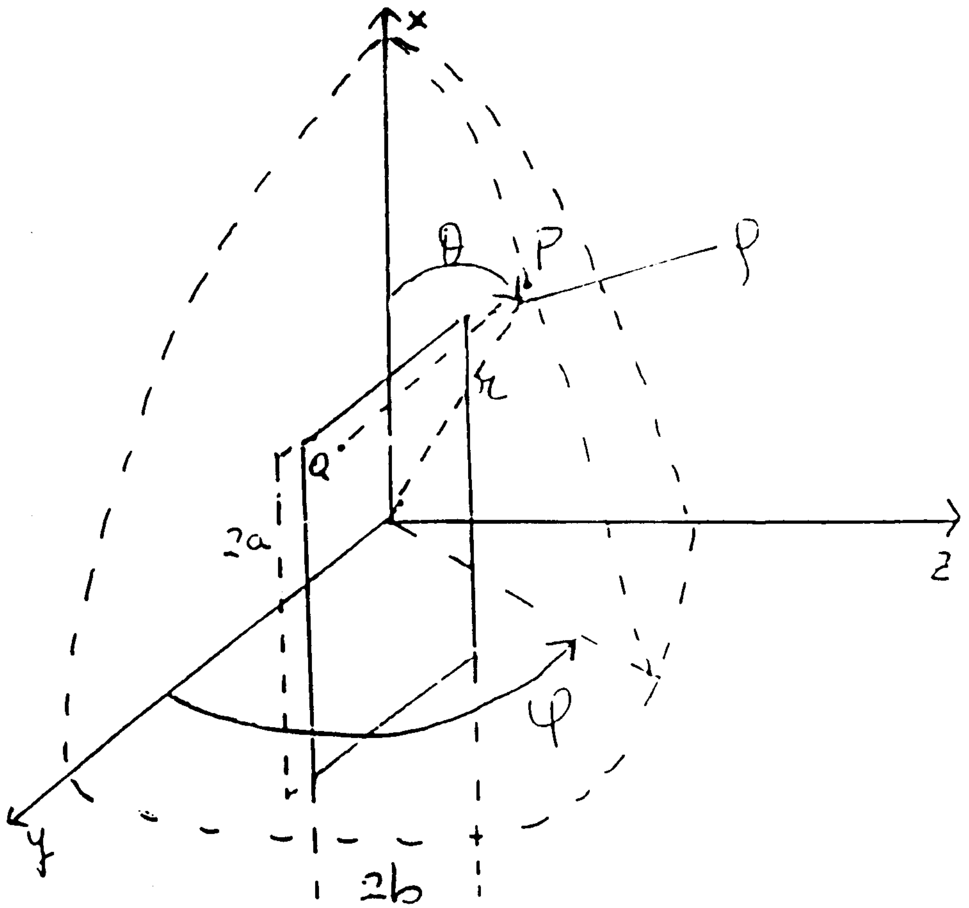
$$\underline{J}_{ms} dx\, dy = 2\, \underline{E}_0 \times \underline{i}_z\, dx\, dy = 2\, E_0\, dx\, dy\, \underline{i}_x$$

A grande distanza:

$$\boxed{\begin{aligned} dH_\varphi &= j \frac{E_0 dx\, dy}{\zeta \lambda \rho} e^{-jk\rho} \operatorname{sen} \theta \\ dE_\theta &= -\zeta\, dH_\varphi \qquad z \geq 0 \end{aligned}}$$

$$H_\theta = j\sqrt{\frac{\varepsilon}{\mu}}\,\frac{I_m \Delta x}{2\lambda r}\,e^{-jkr}\,\mathrm{sen}\,\theta$$

$$\mathbf{E}_\varphi = -\zeta\, H_\theta \qquad \boxed{I_m dx = J_{ms} dx\, dy}$$



$$P \equiv \begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta\cos\varphi \\ z = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi \end{cases}$$

$Q\,(x\,,\,y\,,\,0)$

$$\overline{PQ} = \sqrt{\left(r\cos\theta - x\right)^2 + \left(r\,\mathrm{sen}\,\theta\cos\varphi - y\right)^2 + \left(r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi\right)^2} =$$

$$= r - x\cos\theta - y\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi = \rho$$

$$H_\theta = j\int_{-a}^{+a}\int_{-b}^{+b}\frac{E_0}{\zeta\,\lambda\,\rho}\,e^{-jk\rho}\,\mathrm{sen}\,\theta\,dx\,dy =$$

$$= j\frac{e^{-jkr}}{\zeta\,\lambda r}\,\mathrm{sen}\,\theta\int_{-a}^{+a}e^{jkx\cos\theta}\,dx\int_{-b}^{+b}\left(E_0\left(x,y\right)e^{jky\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi}\right)dy$$

$$E_\varphi = -\zeta\,H_\theta$$

Se $2b << \lambda$ $E_0(x, y) = E_0\,(x)$

$$e^{jky\,\mathrm{sen}\theta\,\mathrm{sen}\varphi} \approx 1$$

$$\left\{ H_\theta = j\frac{2b e^{-jkr}}{\zeta \lambda r} \operatorname{sen}\theta \int_{-a}^{+a} E_0(x) e^{jkx\cos\theta}\, dx \right.$$

$$E_\varphi = -\zeta H_\theta$$

per d.m.c.,

$$H_\theta = j\frac{I_m \Delta z}{2\lambda\zeta r} e^{-jkr} \operatorname{sen}\theta$$

$$\downarrow$$

$$I_m \Delta z = 4b \int_{-a}^{a} E_0(x) e^{jkx\cos\theta}\, dx$$

per z ≤ 0